Instability of Meridional Axial System in $f(R)$ Gravity

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Abstract

We analyze dynamical instability of non-static reflection axial stellar structure by taking into account generalized Euler's equation in metric $f(R)$ gravity. Such an equation is obtained by contracting Bianchi identities of usual anisotropic and effective stress-energy tensors, which after using radial perturbation technique gives modified collapse equation. In the realm of $R+\epsilon R^n$ gravity model, we investigate instability constraints at Newtonian and post-Newtonian approximations. We find that instability of meridional axial self-gravitating system depends upon static profile of structure coefficients while $f(R)$ extra curvature terms induce stability to the evolving celestial body.

Keywords: Axial symmetry; Relativistic fluids; Stability; Modified gravity.
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1 Introduction

General relativity (GR) is believed as a remarkable effort in mathematical physics to analyze gravitational effects of stellar relativistic interiors. Several interesting consequences coming from cosmic microwave background, observational ingredients of Supernovae Ia and its cross-juxtaposition with

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foreground stellar galactic distributions [1]-[3] have made revolution thereby introducing new research window. In this realm, many astrophysicists found GR modifications as helpful to explore unknown aspects of cosmic gravitational dynamics. The $f(R)$ gravity [4] is among extended gravity theories obtained by replacing Ricci invariant with its generic function $f(R)$ in the Einstein-Hilbert action.

Anisotropic effects are leading paradigms in addressing the evolutionary mechanisms of celestial imploding models. Herrera and Santos [5] reviewed contributions of locally anisotropic fluid arrangements on the dynamical phases of collapsing shear and shear-free compact objects. Di Prisco *et al.* [6] investigated dynamical properties of anisotropic spherical matter distribution and found that little fluctuations of pressure anisotropy lead to system cracking. Sharif and his collaborators [7] analyzed effects of anisotropy on the dynamical properties of spherical as well as non-spherical dense relativistic distributions and found much complicated system phases due to the presence of anisotropy. Sunzu *et al.* [8] studied analytical models of spherical anisotropic interiors and found that anisotropic effects provide a broader platform to discuss various forms of stellar relativistic systems. Recently, we [9] explored the dynamical features of anisotropic relativistic interiors.

The spinning stellar distributions indicate direct relevance of anisotropy with gravitational evolution in which gravitational radiations cause vorticity within observer congruences. Vorticity represents rotation of neighboring fluid about an observer moving with relativistic matter distributions relative to an inertial frame. Herrera *et al.* [10] argued that such vorticity seeds from the existence of super-energy flow which may have direct relevance with super-Poynting vector. Bonnor [11] found electromagnetic energy flow in a relativistic compact distribution by formulating a relationship between super-Poynting vector and vorticity. Korunur *et al.* [12] calculated various kinematical variables like angular momentum, energy and momentum of matter configurations associated with an axially symmetric scalar field. Li [13] explored superradiant instability of rotating compact relativistic objects in higher dimensional theory and found unstable configurations against scalar field perturbations. Recently, Herrera *et al.* [14] presented a formal analysis of gravitational radiations within anisotropic non-static reflection axial symmetric source and existence of super-energy flow linked with matter vorticity.

Stability analysis of self-gravitating stellar systems in GR as well as modified gravity have attracted many researchers for last few years. The study of different collapsing celestial models with extra degrees of freedom has great

significance to explore late-time cosmological evolution. Chandrasekhar [15] discussed instability constraints for spherical symmetric relativistic geometry coupled with ideal matter configurations using ratio of specific heats known as stiffness parameter, Γ_1. Herrera *et al.* [16] investigated stability regions for radiating collapsing stellar objects and concluded that dissipation vector tends to move the systems towards stable configurations. Chan *et al.* [17] studied remarkable effects of shearing viscosity and anisotropy on the instability constraints at Newtonian (N) and post-Newtonian (pN) eras.

Cai [18] discussed dynamical properties and structure formation of dense matter relativistic configurations in modified gravity by assigning zero, negative, and positive values of constant curvature. Bamba *et al.* [19] performed dynamical analysis of a collapsing relativistic stellar system and claimed that invoking of $R^\alpha(1 < \alpha \leq 2)$ corrections could helps to present a viable and singularity free model. Myung *et al.* [20] performed stability analysis of sphericalstellar structure with constant Ricci invariant background in metric $f(R)$ gravity via perturbation scheme and noticed relatively stable distributions under specific constraints. Moon *et al.* [21] extended these consequences with negative cosmological constant environment and calculated limits for the stability of relativistic systems.

Capozziello *et al.* [22] explored dynamical evolution of relativistic collapsing spherical interior in $f(R)$ gravity by evaluating extended form of Poisson and Boltzmann equations. De Laurentis and Capozziello [23] discussed instability issue of stellar interior at N approximation with $f(R)$ extra degrees of freedom and also studied axisymmetric black hole models. Astashenok *et al.* [24] investigated evolution of self-gravitating systems and found relatively more massive and supergiant dense configurations due to $f(R)$ gravity corrections. Farinelli *et al.* [25] discussed dynamical properties of stellar systems in the presence of $f(R)$ corrections and found that higher degree terms tend to mollify collapsing process. Sharif and his collaborators [26, 27] studied instability constraints for restricted class of axially symmetric spacetime by means of adiabatic index/stiffness parameter.

The present paper aims to extend our previous work [27] of stability analysis by taking reflection effects in non-static axial symmetric anisotropic source with ϵR^n extra degrees of freedom. In the present paper, we develop instability regions for anisotropic meridional axisymmetric source with $R + \epsilon R^n$ background. The inclusion of ϵR^n correction in our analysis seeds from the fact that this correspond to the various eras of the cosmic history thereby helping to explain the gravitational dynamics during inflationary as

well as late-time accelerating regimes. Furthermore, the addition of meridional effects in stellar system causes a flow of gravitational energy due to existence of vorticity tensor in the analysis.

The paper has the following format. Section **2** deals with kinematical formulations of comoving meridional axial symmetric geometry coupled with anisotropic matter configurations. The meridional effects in stellar system causes a flow of gravitational energy due to existence of vorticity tensor. We present $f(R)$ dark source components and set of dynamical equations with reflection axial degrees of freedom. In section **3**, we discuss viable $f(R)$ model and use perturbation method to develop collapse equation. Section **4** explores instability constraints. Finally, we summarize our results in the last section.

2 Anisotropic Source and Field Equations

The extended configuration of Einstein-Hilbert action is

$$S_{f(R)} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R) + S_M, \tag{1}$$

where $\kappa,\ f(R),\ S_M,\ T_{\alpha\beta}$ are coupling constant, matter action, a non-linear Ricci curvature function and usual stress energy tensor, respectively. The variation of above action with respect to $g_{\alpha\beta}$ provides the field equations

$$f_R R_{\alpha\beta} - \nabla_\alpha\nabla_\beta f_R - g_{\alpha\beta}\left(\frac{1}{2}f - \Box f_R\right) = \kappa T_{\alpha\beta}, \tag{2}$$

where $\Box,\ \nabla_\alpha,\ f_R$ are D'Alembert, covariant derivative and $\frac{df}{dR}$ operators, respectively. Equation (2) can be written in terms of Einstein tensor as

$$G_{\alpha\beta} = \frac{\kappa}{f_R}(\overset{(D)}{T}_{\alpha\beta} + T_{\alpha\beta}), \tag{3}$$

where

$$\overset{(D)}{T}_{\alpha\beta} = \frac{1}{\kappa}\left\{\frac{R}{2}\left(\frac{f}{R} - f_R\right)g_{\alpha\beta} - \Box f_R g_{\alpha\beta} + \nabla_\alpha\nabla_\beta f_R\right\}, \tag{4}$$

is the stress energy tensor indicating $f(R)$ contribution in the dynamics of relativistic systems. We take axially symmetric metric characterizing reflection effects [14]

$$ds^2 = -A^2(t,r,\theta)dt^2 + 2L(t,r,\theta)dtd\theta + B^2(t,r,\theta)(dr^2 + r^2d\theta^2) + C^2(t,r,\theta)d\phi^2, \tag{5}$$

with locally anisotropic fluid configuration

$$T_{\alpha\beta}=(\mu+P)V_\alpha V_\beta+Pg_{\alpha\beta}+\frac{1}{3}(\Pi_{II}+2\Pi_I)(K_\alpha K_\beta-\frac{1}{3}h_{\alpha\beta})+\frac{1}{3}(\Pi_I+2\Pi_{II})\\ \times(N_\alpha N_\beta-\frac{1}{3}h_{\alpha\beta})+\Pi_{KN}(K_\alpha N_\beta+K_\beta N_\alpha), \tag{6}$$

where $\mu,\ P,\ \Pi_I,\ \Pi_{II},\ \Pi_{KL}$ and $h_{\alpha\beta}$ are the fluid energy density, pressure, anisotropic scalars and projection tensor, respectively. The matter four velocity, V_α, and spacelike vectors $S_\alpha,\ K_\alpha$ and N_α in comoving coordinates are

$$V^\alpha=\frac{1}{A}\delta^0_\alpha,\ V_\alpha=-A\delta^0_\alpha+\frac{L}{A}\delta^2_\alpha,\ S_\alpha=C\delta^3_\alpha,\ K_\alpha=B\delta^1_\alpha,\ N_\alpha=\frac{\sqrt{\Delta}}{A}\delta^2_\alpha, \tag{7}$$

where $\Delta=(ABr)^2+L^2$, which obey the following constraints

$$K^\alpha N_\alpha=K^\alpha S_\alpha=S^\alpha N_\alpha=V_\alpha K^\alpha=V^\alpha N_\alpha=V^\alpha S_\alpha=0,\\ K_\alpha K^\alpha=N_\alpha N^\alpha=S_\alpha S^\alpha=-V^\alpha V_\alpha=1.$$

The fluid pressure and its anisotropic scalars can be expressed alternatively in terms of projection tensor and spacelike vectors, respectively as

$$P=\frac{1}{3}h^{\alpha\beta}T_{\alpha\beta},\quad \Pi_I=(2K^\alpha K^\beta-S^\alpha S^\beta-N^\alpha N^\beta)T_{\alpha\beta},\quad \Pi_{KN}=K^\alpha N^\beta T_{\alpha\beta},\\ \Pi_{II}=(2N^\alpha N^\beta-K^\alpha K^\beta-S^\alpha S^\beta)T_{\alpha\beta}.$$

The non-zero components of effective stress energy tensor (4) are

$$\overset{(D)}{T^{\alpha\beta}}=\begin{bmatrix} V_1+W_1 & X_1+Y_1 & X_3+Y_3 & 0\\ X_1+Y_1 & V_2+W_2 & X_2+Y_2 & 0\\ X_3+Y_3 & X_2+Y_2 & V_3+W_3 & 0\\ 0 & 0 & 0 & V_4+W_4\end{bmatrix}, \tag{8}$$

where dark source $f(R)$ terms $V_i,\ W_i$ and $X_j,\ Y_j$ are diagonal and non-diagonal components of effective energy-momentum tensor (4), respectively, in which W_i and Y_j incorporate axial reflection effects with $f(R)$ extra degrees of freedom. By choosing X_j and Y_j equal to zero along with $\Delta\rightarrow A^2B^2r^2$, higher curvature terms of restricted axisymmetric metric can be found. However, the inclusion of these terms along with anisotropic in usual stress energy

tensor ensure the propagation of gravitational radiations in the environment [28].

The kinematical quantity controlling local spinning motion of anisotropic matter distributions is the vorticity tensor which for meridional axially symmetric metric can be expressed in terms of K_α and N_α as

$$\Omega_{\alpha\beta}=\Omega(K_\beta N_\alpha-N_\beta K_\alpha),$$

where

$$\Omega=\frac{L}{2B\sqrt{\Delta}}\left(\frac{L'}{L}-\frac{2A'}{A}\right) \tag{9}$$

is known as vorticity scalar. Here prime stand for $\frac{\partial}{\partial r}$. There exists only one independent non-zero vorticity component along $r\theta$ direction. The existence of vorticity scalar is directly related to the existence of reflection effects of axisymmetric spacetime as it is controlled by non-diagonal structure coefficient, L. Thus if one takes $\Omega=0$ over the dynamical evolution of axial anisotropic spacetime, this imparts null value to non-diagonal scale factor whose dynamics has already been discussed [27].

In order to evaluate dynamical evolution equations for axially symmetric relativistic celestial body with $f(R)$ background, we consider

$$|T^{\alpha\beta}+\overset{(D)}{T^{\alpha\beta}}|_{;\beta}=0,$$

which yields

$$\begin{aligned}&\dot{\mu}-\mu\left[\frac{\dot{B}}{B}+\frac{\dot{C}}{C}+\frac{1}{\Delta}\left(r^2A\dot{A}B^2+L\dot{L}+r^2A^2B\dot{B}\right)\right]+\frac{AB^2(\mu+P)}{\Delta}\left[r^2\left(\frac{2\dot{B}}{B}\right.\right.\\&\left.\left.+\frac{\dot{C}}{C}\right)+\frac{L^2}{A^2B^2}\left(\frac{\dot{B}}{B}-\frac{\dot{A}}{A}+\frac{\dot{L}}{L}+\frac{\dot{C}}{C}\right)\right]+\frac{\Pi_I}{3A}\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)+\frac{\Pi_{II}}{3\Delta}\left\{AB^2r^2\left(\frac{\dot{B}}{B}\right.\right.\\&\left.\left.-\frac{\dot{C}}{C}\right)+\frac{L^2}{A}\left(\frac{\dot{L}}{L}-\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right)\right\}+\left(\frac{B^2r^2A\dot{A}}{\Delta}+\frac{\dot{C}}{C}\right)V_1+D_0(t,r,\theta)=0,\end{aligned} \tag{10}$$

$$P'+\frac{2}{9}\left(2\Pi_I'+\Pi_{II}'\right)+\left[P+\frac{2}{9}\left(2\Pi_I+\Pi_{II}\right)\right]\left[\frac{C'}{C}+\frac{3LL'}{2\Delta}+\frac{r^2A^2B^2}{\Delta}\left(\frac{A'}{A}\right.\right.$$

$$+\frac{2B'}{B}+\frac{2}{r}-\frac{(rB)'}{rB}\Big)\Big]-\frac{r^2AB^5}{\Delta^{3/2}}\Big[\Pi_{KN,\theta}-\Big\{\frac{A_\theta}{A}+\frac{6B_\theta}{B}+\frac{C_\theta}{C}+\frac{4r^2A^2B^2}{\Delta}$$
$$\times\left(\frac{A_\theta}{A}+\frac{B_\theta}{B}\right)+\frac{4LL_\theta}{\Delta}\Big\}\Pi_{KL}\Big]+\frac{\mu r^4A^4B^4}{\Delta^2}\left(B\dot{B}+\frac{A'}{A}-\frac{LA_\theta}{r^2AB^2}\right)-\Big(\frac{(rB)'}{rB}$$
$$+\frac{L}{2L'}\Big)\frac{\mu r^2A^2L^2B^2}{\Delta^2}+\left(\frac{3\dot{B}}{B}+\frac{r^2B^2A\dot{A}}{\Delta}+\frac{\dot{C}}{C}\right)X_1+D_1(t,r,\theta)=0, \tag{11}$$
$$\frac{\mu r^2A^2B^2L}{\Delta^2}\Big[\frac{\dot{\mu}}{\mu}+\frac{\dot{A}}{A}+\frac{3\dot{B}}{B}+\frac{\dot{L}}{L}+\frac{\dot{C}}{C}+\frac{1}{r^2B^2}\left(\frac{\mu_\theta}{\mu}+\frac{2L_\theta}{L}+\frac{2A_\theta}{A}\right)+\frac{1}{\Delta}\Big\{4r^2A^2$$
$$\times\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right)-\frac{4\dot{L}}{L}-LA^2\left(\frac{5A_\theta}{A}+\frac{2B_\theta}{B}\right)+r^2A^2B^2\left(\frac{\dot{L}}{L}+\frac{\dot{B}}{B}\right)+\frac{r^2A^3B^2A_\theta}{L}\Big\}$$
$$-\frac{4L^2L_\theta\Delta}{r^2B^2}\Big]+\frac{\mu A^2L^2}{\Delta^2}\left\{\frac{B_\theta}{B}+\frac{C_\theta}{C}-\frac{r^2BL\dot{B}}{\Delta}\right\}-\frac{r^3AB^3\Pi_{KN}}{\Delta^{\frac{3}{2}}}\Big[\frac{\Pi'_{KN}}{\Pi_{KN}}+\frac{3}{r}$$
$$+\frac{4B'}{B}+\frac{A'}{A}+\frac{C'}{C}+\frac{3}{\Delta}\left\{\frac{LL'}{2}+r^2A^2B^2\left(\frac{3}{r}+\frac{2A'}{A}+\frac{3B'}{B}\right)\right\}+\frac{7LL'}{2\Delta}\Big]$$
$$+\frac{1}{\Delta}\left\{P+\frac{2}{9}(\Pi_I+2\Pi_{II})\right\}\Big[\frac{r^2A^2B^2}{\Delta}\Big\{(2A^2+A)\left(\frac{A_\theta}{A}+\frac{B_\theta}{B}\right)-\frac{L\dot{B}}{B}$$
$$+\frac{2AB_\theta}{B}\Big\}+2AA_\theta+\frac{A^2C_\theta}{C}-\frac{r^2BL\dot{L}}{\Delta}-\frac{2A^2LL_\theta}{\Delta}-\frac{L\dot{B}}{B}\Big]-\frac{P}{C\Delta}(L\dot{C}$$
$$+A^2C_\theta)+\frac{A^2}{\Delta}\left\{P_\theta+\frac{2}{9}(\Pi_{I,\theta}+2\Pi_{II,\theta})\right\}+D_2(t,r,\theta)=0, \tag{12}$$

where D_0, D_1 and D_2 are $f(R)$ corrections given in Appendix **A**. Here over dot and subscript θ stand for $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial\theta}$, respectively. The second of the above equations is known as generalized Euler equation.

3 $f(R)$ Model and Perturbation Scheme

Many inflationary models in the early universe are established on scalar fields coming in supergravity and superstring theories. The first model of inflation was suggested by Starobinsky which deals with conformal anomaly in quantum gravity [29] given by [30]

$$f(R)=R+\epsilon R^n, \tag{13}$$

where n can be positive or negative integer. This model explains the present universe acceleration due to the presence of dark energy and can serve as power-law inflation, i.e. exponential expansion and ordinary inflation incorporating minimally coupled scalar field. Here $\epsilon \sim \frac{1}{M^{2n-2}} > 0$ for $n > 0$ and M has the mass dimensions. Since $f(R)$ gravity can be used as an alternative for dark matter [31] in addition to dark energy at cluster as well as stellar scales, so this model with $n = 2$ was claimed both as dark matter model with $\epsilon = \frac{1}{6M^2}$ [32] and as dark energy. The value of M is chosen to be $2.7 \times 10^{-12} GeV$ along with $\epsilon \leq 2.3 \times 10^{22} Ge/V^2$ for dark matter cosmology [33]. All GR solutions can be found by taking limit $f(R) \to R$.

Here, we use perturbation method [16, 17] to explore modified collapse equation for meridional axially symmetric anisotropic geometry. For very small values of perturbation parameter α with $0 < \alpha \ll 1$, we take effects up to $O(\alpha)$. We first suppose that the system is in hydrostatic equilibrium at $t = 0$, however on departing from this state, the system depends upon the same time dependence factor $T(t)$ on all its structure coefficients. The structure and matter variables can be perturbed as follows

$$\begin{aligned} \mathcal{S}(t, r, \theta) &= \mathcal{S}_0(r, \theta) + \alpha T(t) s(r, \theta), & (14) \\ \mathcal{M}(t, r, \theta) &= \mathcal{M}_0(r, \theta) + \alpha \bar{m}(t, r, \theta), & (15) \end{aligned}$$

where $\mathcal{S}$ represents perturbation method applicable on structural co-efficients of Eq.(5), i.e., $A,\ B,\ C,\ L$ and on Ricci scalar, R which after perturbation denotes s as $a,\ b,\ c,\ l$ and e, respectively. Equation (15) indicates perturbation method of matter variables (these matter variables are taken from Eq.(6)). Thus the allocation of $\mathcal{M}$ will be $\mu,\ P,\ \Pi_a,\ a = 1, 2, 3$ and the corresponding perturbed quantities will be represented by placing bar over that. However, the perturbation technique for $f(R)$ model is given as follows

$$\begin{aligned} f(t, r) &= [R_0(r) + \epsilon R_0^n)] + \alpha T(t) e(r) \left[1 - \epsilon n R_0^{n-1}\right], & (16) \\ f_R(t, r) &= 1 + \epsilon n R_0^{n-1} + \alpha T(t) e(r) n\epsilon(n-1) R_0^{n-2}, & (17) \end{aligned}$$

where R_0 represents static distribution of Ricci scalar. Using Eqs.(14)-(17), the first of dynamical equations satisfies trivially, while rest of dynamical equations (11) and (12) at $t = 0$ give

$$P_0' + \frac{2}{9}\left(2\Pi_{I0}' + \Pi_{II0}'\right) + \left[P_0 + \frac{2}{9}\left(2\Pi_{I0} + \Pi_{II0}\right)\right]\left[\frac{C_0'}{C_0} + \frac{3L_0 L_0'}{2\Delta_0} + \frac{r^2 A_0^2 B_0^2}{\Delta_0}\right.$$

$$\times \left(\frac{A_0'}{A_0}+\frac{1}{r}\right)\Bigg] - \frac{r^3A_0B_0^5}{\Delta_0^{\frac{3}{2}}}\Pi_{KN0,\theta} - \frac{r^3A_0B_0^5}{\Delta_0^{\frac{3}{2}}}\left\{\frac{A_{0\theta}}{A}+\frac{6B_{0\theta}}{B_0}+\frac{C_{0\theta}}{C_0}+\frac{4L_0L_{o\theta}}{\Delta_0}\right.$$
$$\left.+\frac{4r^2A_0^2B_0^2}{\Delta_0}\left(\frac{A_{0\theta}}{A_0}+\frac{B_{0\theta}}{B_0}\right)\right\} + \frac{\mu r^4A_0^4B_0^4}{\Delta_0^2}\left(\frac{A_0'}{A_0}-\frac{L_0A_{0\theta}}{r^2A_0B_0^2}\right) - \frac{\mu_0r^2A_0^2L_0^2B_0^2}{\Delta_0^2}$$
$$\left(\frac{L_0}{2L_0'}+\frac{1}{r}+\frac{B_0'}{B_0}\right) + D_{1S} = 0, \tag{18}$$
$$\frac{\mu_0r^2A_0^2B_0^2L_0}{\Delta^2}\left[+\frac{1}{r^2B_0^2}\left(\frac{\mu_{0\theta}}{\mu_0}+\frac{2L_{0\theta}}{L_0}+\frac{2A_{0\theta}}{A_0}\right)-\frac{1}{\Delta_0}\left\{L_0A_0^2\left(\frac{5A_{0\theta}}{A_0}+\frac{2B_{0\theta}}{B_0}\right)\right.\right.$$
$$\left.\left.-\frac{r^2A_0^3B_0^2A_{0\theta}}{L_0}\right\}-\frac{4L_0^2L_{0\theta}\Delta_0}{r^2B_0^2}\right]+\frac{\mu A_0^2L_0^2}{\Delta_0^2}\left\{\frac{B_{0\theta}}{B_0}+\frac{C_{\theta 0}}{C}\right\}-\frac{r^3A_0B_0^3\Pi_{KN0}}{\Delta_0^{3/2}}\left[\frac{3}{r}\right.$$
$$+\frac{\Pi_{KN0}'}{\Pi_{KN0}}+\frac{4B_0'}{B_0}+\frac{A_0'}{A_0}+\frac{C_0'}{C_0}+\frac{3}{\Delta_0}\left\{\frac{L_0L_0'}{2}+r^2A_0^2B_0^2\left(\frac{3}{r}+\frac{2A_0'}{A_0}+\frac{3B_0'}{B_0}\right)\right\}$$
$$\left.+\frac{7L_0L_0'}{2r\Delta_0}\right]+\frac{1}{\Delta_0}\left\{P_0+\frac{2}{9}(\Pi_{I0}+2\Pi_{II0})\right\}\left[\frac{r^2A_0^2B_0^2}{\Delta_0}\left\{(2A_0^2+A_0)\left(\frac{A_{0\theta}}{A_0}+\frac{B_{0\theta}}{B_0}\right)\right.\right.$$
$$\left.\left.+\frac{2A_0B_{0\theta}}{B_0}\right\}+2A_0A_{0\theta}+\frac{A_0^2C_{0\theta}}{C_0}-\frac{2A_0^2L_0L_{0\theta}}{\Delta_0}\right]+A_0^2C_{0\theta}+\frac{A_0^2}{\Delta_0}\left\{P_{0\theta}+\frac{2}{9}(\Pi_{I0\theta}\right.$$
$$\left.+2\Pi_{II0\theta})\right\}+D_{2S}=0. \tag{19}$$

The static $f(R)$ contribution of second and third conservation equations are denoted by D_{2S} and D_{3S}, respectively and can be calculated very easily from Eqs.(A2) and (A3) after using perturbation method. Using Eqs.(14)-(17), the non-static perturbed axial dynamical equation (10) will take the form

$$\dot{\mu}+\left[\mu_0\left\{\frac{b}{B_0}+\frac{c}{C_0}+\frac{1}{\Delta_0}\left(r^2aA_0^2B_0^2+lL_0+r^2bB_0L_0\right)\right\}+(\mu_0+P_0)\frac{A_0^2B_0^2}{\Delta_0^2}\right.$$
$$\times\left\{r^2\left(\frac{2b}{B_0}+\frac{2c}{C_0}\right)+\frac{L_0^2}{A_0^2B_0^2}\left(\frac{b}{B_0}+\frac{l}{L_0}-\frac{a}{A_0}+\frac{c}{C_0}\right)\right\}+\frac{\Pi_{I0}}{3}\left(\frac{b}{B_0}-\frac{c}{C_0}\right)$$
$$\left.+\frac{\Pi_{II0}}{3\Delta_0}\left\{r^2A_0^2B_0^2\left(\frac{b}{B_0}-\frac{c}{C_0}\right)+L_0^2\left(\frac{l}{L_0}-\frac{a}{A_0}-\frac{c}{C_0}\right)\right\}+D_3(r,\theta)\right]\dot{T}=0,$$

where D_3 represents $f(R)$ corrections which can be obtained from expressions $g(t,r,\theta)$ and $h(t,r,\theta)$ given in Appendix **A**. Substituting Eq.(13) in Eq.(8) and then employing perturbation method, one can obtain $f(R)$ dynamical quantities, $V_i,\ W_i,\ X_j,\ Y_j$ whose values upon substitution in Eqs.(A4) and (A5) yield D_3 such that $g(t,r,\theta)+h(t,r,\theta)=D_3\dot{T}$. Integration of the above

equation gives

$$\bar{\mu} = -\chi(r,\theta)T, \tag{20}$$

where

$$\chi = \left[\mu_0\left\{\frac{b}{B_0}+\frac{c}{C_0}+\frac{1}{\Delta_0}\left(r^2aA_0^2B_0^2+lL_0+r^2bB_0L_0\right)\right\}+(\mu_0+P_0)\frac{A_0^2B_0^2}{\Delta_0^2}\right.$$
$$\times\left\{r^2\left(\frac{2b}{B_0}+\frac{2c}{C_0}\right)+\frac{L_0^2}{A_0^2B_0^2}\left(\frac{b}{B_0}+\frac{l}{L_0}-\frac{a}{A_0}+\frac{c}{C_0}\right)\right\}+\frac{\Pi_{I0}}{3}\left(\frac{b}{B_0}-\frac{c}{C_0}\right)$$
$$\left.+\frac{\Pi_{II0}}{3\Delta_0}\left\{r^2A_0^2B_0^2\left(\frac{b}{B_0}-\frac{c}{C_0}\right)+L_0^2\left(\frac{l}{L_0}-\frac{a}{A_0}-\frac{c}{C_0}\right)\right\}+D_3(r,\theta)\right].$$

Now, we evaluate $t\theta$ component of metric $f(R)$ field equations (3) and then using perturbation scheme along with some manipulations, it follows that

$$\varrho_1\ddot{T}+\varrho_2\dot{T}+\varrho_3T=0, \tag{21}$$

where quantities ϱ_i contain combinations of meridional axial geometric functions as well as $R+\epsilon R^n$ corrections, depending upon r and θ coordinates and are assumed positive. More specifically, these quantities incorporate non-perturbed as well as perturbed terms. There exist oscillating as well as non-oscillating solutions of the above equation which represent unstable as well as stable models of evolving relativistic stellar systems, respectively. We confine ourselves to obtain solutions for collapsing relativistic system. Thus we limit our perturbation parameters, $a,\ b,\ c,\ e$ and l to be positive definite quantities for which we obtain $\omega^2>0$. In this context, the solution of Eq.(21) is given by

$$T(t)=-\exp(\omega t), \quad \text{where} \quad \omega^2=\frac{-\varrho_2+\sqrt{\varrho_2^2-4\varrho_1\varrho_3}}{2\varrho_1}. \tag{22}$$

Using the perturbation technique, the non-static distributions of Eq.(11), after using Eq.(22), are written as

$$\frac{1}{B_0^2}\left\{\bar{P}'+\frac{2}{9}(2\bar{\Pi}_I'+\bar{\Pi}_{II}')\right\}+\frac{1}{B_0^2}\left\{\bar{P}+\frac{2}{9}(2\bar{\Pi}_I+\bar{\Pi}_{II})\right\}\left\{\frac{C_0'}{C_0}+\frac{3L_0L_0'}{2\Delta_0}+\left(\frac{1}{r}\right.\right.$$
$$\left.\left.+\frac{A_0'}{A_0}\right)\frac{r^2A_0^2B_0^2}{\Delta_0}\right\}-\frac{r^3A_0B_0^3}{\Delta_0^{3/2}}\bar{\Pi}_{KN,\theta}-\bar{\Pi}_{KN}\frac{r^3A_0B_0^3}{\Delta_0^{3/2}}\left\{\frac{A_{0\theta}}{A_0}+\frac{6B_{0\theta}}{B_0}+\frac{4L_0L_{0\theta}}{\Delta_0}\right.$$

$$
+\frac{C_{0\theta}}{C_0}+\frac{4r^2A_0B_0^2}{\Delta_0}\left(\frac{A_{0\theta}}{A_0}+\frac{B_{0\theta}}{B_0}\right)\Bigg\}+\frac{\bar{\mu}r^4A_0^4}{\Delta_0^2}\left(\frac{A_0'}{A_0}-\frac{L_0A_{0\theta}}{r^2A_0B_0^2}\right)-\bar{\mu}L_0^2r^2\frac{A_0^2}{\Delta_0^2}
\times\left(\frac{L_0'}{2L_0}+\frac{B_0'}{B_0}\right)-\frac{2b}{B_0^3}\left\{{P_0}'+\frac{2}{9}(2\Pi_{I0}'+\Pi_{II0}')\right\}T+\frac{r^3A_0B_0^3}{\Delta_0^{3/2}}\Pi_{KN0,\theta}\left(\frac{a}{A_0}
+\frac{3b}{B_0}-\frac{3d}{\Delta_0}\right)T+\frac{lL_0X_{10}}{\Delta_0}+\frac{r^2A_0^2bB_0}{\Delta_0}X_{30}+[\Upsilon+\Phi+\zeta]T=0, \tag{23}
$$

where Υ, ζ and Φ are mentioned in Appendix **A**. The quantity controlling the reflection degrees of freedom along with $f(R)$ corrections of an axisymmetric celestial body is Υ. However, the expression Φ incorporates gravitational contribution due to $f(R)$ gravity, while ζ is the remaining part of non-static perturbed generalized Euler equation holding usual Einstein gravity effects.

In view of second law of thermodynamics, we can link perturbed anisotropic quantities with energy density by an equation of state as [34]

$$
\bar{P}_i=\Gamma_1\frac{P_{i0}}{\mu_0+P_{i0}}\bar{\mu}, \tag{24}
$$

where Γ_1 is a fluid stiffness parameter also known as adiabatic index. This measures pressure variations of matter configurations with respect to energy density. In our analysis, Γ_1 will be treated as a constant identity. Using Eqs.(20) and (24), we have

$$
\bar{\Pi}_{KN}=-\Gamma_1\frac{\Pi_{KN0}}{\mu_0+\Pi_{KN0}}\chi T,\quad \bar{P}=-\Gamma_1\frac{P_0}{\mu_0+P_0}\chi T,
$$
$$
\bar{\Pi}_{I}=-\Gamma_1\frac{\Pi_{I0}}{\mu_0+\Pi_{I0}}\chi T,\quad \bar{\Pi}_{II}=-\Gamma_1\frac{\Pi_{II0}}{\mu_0+\Pi_{II0}}\chi T.
$$

Using Eq.(20) as well as the above relations in Eq.(23), we obtain

$$
-\frac{1}{B_0^2}\Gamma_1\phi'T-\frac{1}{B_0^2}\Gamma_1\phi T\left\{\frac{C_0'}{C_0}+\frac{3L_0L_0'}{2\Delta_0}+\left(\frac{1}{r}+\frac{A_0'}{A_0}\right)\frac{r^2A_0^2B_0^2}{\Delta_0}\right\}-\frac{r^3A_0B_0^3}{\Delta_0^{3/2}}\Gamma_1T
\times\left(\frac{\Pi_{KN0}\chi}{\mu_0+\Pi_{KN0}}\right)_\theta-\frac{\Pi_{KN0}\chi}{\mu_0+\Pi_{KN0}}\frac{r^3A_0B_0^3}{\Delta_0^{3/2}}\left\{\frac{A_{0\theta}}{A_0}+\frac{4r^2A_0B_0^2}{\Delta_0}\left(\frac{A_{0\theta}}{A_0}+\frac{B_{0\theta}}{B_0}\right)\right.
\left.+\frac{C_{0\theta}}{C_0}+\frac{6B_{0\theta}}{B_0}+\frac{4L_0L_{0\theta}}{\Delta_0}\right\}-T\frac{\chi r^4A_0^4}{\Delta_0^2}\left(\frac{A_0'}{A_0}-\frac{L_0A_{0\theta}}{r^2A_0B_0^2}\right)+T\chi L_0^2r^2\frac{A_0^2}{\Delta_0^2}\left(\frac{L_0'}{2L_0}
+\frac{B_0'}{B_0}\right)-\frac{2b}{B_0^3}\left\{{P_0}'+\frac{2}{9}(2\Pi_{I0}'+\Pi_{II0}')\right\}T+\frac{r^3A_0B_0^3}{\Delta_0^{3/2}}\left(\frac{a}{A_0}+\frac{3b}{B_0}-\frac{3d}{\Delta_0}\right)
$$

$$\times \Pi_{KN0,\theta}T + \frac{lL_0X_{10}}{\Delta_0} + \frac{r^2A_0^2bB_0}{\Delta_0}X_{30} + [\Upsilon + \Phi + \zeta]T = 0, \tag{25}$$

where $\phi = \frac{P_0\chi}{(\mu_0+P_0)} + \frac{4\Pi_{I0}\chi}{9(\mu_0+\Pi_{I0})} + \frac{2\Pi_{II0}\chi}{9(\mu_0+\Pi_{II0})}$. The above equation is known as collapse equation of axisymmetric stellar objects characterizing meridional and $f(R)$ extra order degrees of freedom.

4 Instability Regions

Now we proceed to calculate constraints at which meridional axial symmetric stellar systems undergo instability window at both N and pN eras with $f(R)$ background. We also examine the role of stiffness parameter Γ_1 in this scenario. We also reduce our results to previously known limiting cases. The formulation of instability constraints should be compatible with Tolman-Oppenheimer-Volkoff (TOV) equation. Such type of equation constrains the relativistic stellar structure coupled with matter distribution at the phase of static gravitational equilibrium. In this respect, Barausse *et al.* [35] investigated hydrostatic equilibrium phases of relativistic models by obtaining modified version of TOV equation in $f(R)$ gravity. Recently, Astashenok *et al.* [36] calculated extended version of TOV equation with equation of state in the realm of cubic as well as quadratic corrections and found that such an equation can be used to describe viable models of compact objects. Here, we formulate TOV equation that will help us to obtain some limits on fluid-energy density and its derivatives to avoid curvature divergence at the stellar boundary. The 11 and 22 components of metric $f(R)$ field equations, respectively, provide

$$\frac{A'}{A} = \frac{B^2}{\gamma}\left[\frac{\kappa}{f_R}\left(P + \frac{\Pi_I}{3} + \frac{\xi_2}{\kappa B^2}\right) - \frac{\xi_1}{\Delta^2B^2}\right], \tag{26}$$

$$\frac{A_\theta}{A} = \frac{1}{\gamma_1}\left[\frac{\kappa}{f_R}\left\{\frac{\mu L^2}{A^2} + \frac{\Delta}{A^2}\left(P + \frac{2}{9}\left(\Pi_{II} + 2\Pi_I\right)\right) + \frac{\xi_4}{\kappa}\right\} - \frac{\xi_3}{4A^2}\right], \tag{27}$$

where

$$\gamma = \frac{A^2B^2r^2}{\Delta}\left[\frac{16}{\Delta}\left(1 + \frac{rC'}{C} + \frac{rB'}{B}\right) + \frac{rf_R'}{f_R}\right], \tag{28}$$

$$\gamma_1 = \frac{A^2B^2r^2L^2}{\Delta^2}\left(\frac{r^2B^2\dot{C}}{LC} - \frac{2B_\theta}{B}2\frac{2C_\theta}{C}\right) - \frac{r^2B^2A^2f_{R\theta}}{\kappa f_R}\left(\frac{r^2A^2B^2}{\Delta} - \frac{L^2}{\Delta}\right)$$

$$
-\frac{B^4A^4r^4}{\Delta^2}\left(\frac{C_\theta}{C}+\frac{B_\theta}{B}\right), \tag{29}
$$

$$
\xi_1=\Delta^2G_{11}-4\Delta r^2A^2b^2\frac{A'}{A}\left(\frac{4}{r}+\frac{4C'}{C}+\frac{4B'}{B}\right),\quad \xi_2=\overset{(D)}{T}_{11}+\frac{A^2B^2r^2f'_RA'}{\kappa A\Delta},
$$

$$
\xi_3=4\Delta^2G_{22}+4B^4A^4r^4\left(\frac{A_\theta C_\theta}{AC}+\frac{A_\theta B_\theta}{AB}\right)-4A^2B^2r^2L^2\left(\frac{r^2B^2\dot{C}A_\theta}{LCA}-\frac{2A_\theta B_\theta}{AB}\right.
$$

$$
\left.-\frac{2A_\theta C_\theta}{AC}\right),\quad \xi_4=\overset{(D)}{T}_{22}-\frac{A^2B^2r^2f_{R\theta}}{\Delta}\left(\frac{r^2B^2AA_\theta}{\Delta}-\frac{L^2A_\theta}{A\Delta}\right).
$$

The corresponding Misner-Sharp mass function [37] takes the form

$$
m_{tot}=\frac{r^3B}{2}\left(\frac{r^2B^2\dot{B}^2}{\Delta}-\frac{B'}{rB^2}-\frac{2B'}{rB}-\frac{A^2B_\theta^2}{\Delta}-\frac{2LB_\theta\dot{B}}{\Delta}\right). \tag{30}
$$

Using Eq.(30), second and third laws of conservation of usual energy-momentum tensor as well as Eqs.(26) and (27), we obtain TOV equations

$$
-P'=\left[\frac{9\mu r^4A^4\psi_m^4-9r^2A^2\psi_m^2\Delta\{9P+2(\Pi_{II}+2\Pi_I)\}}{9\Delta^2}\right]\frac{\psi_m^2}{\gamma}\left[\frac{\kappa}{f_R}\left(P+\frac{\Pi_I}{3}\right.\right.
$$

$$
\left.\left.+\frac{\xi_2}{\kappa\psi_m^2}\right)-\frac{\xi_1}{\Delta^2\psi_m^2}\right]+\left[\xi_5+\frac{2}{9}(2\Pi_{II}+\Pi_I)\right],
$$

$$
-P_\theta=\left[\frac{-5\mu r^2A^4\psi_m^2L-r^2A^2\psi_m^2\{9P+2(\Pi_{II}+2\Pi_I)\}}{9\Delta^2}\right]\frac{\Delta}{\gamma_1A^2}\left[\frac{\kappa}{f_R}\left\{\frac{\mu L^2}{A^2}\right.\right.
$$

$$
\left.\left.+\frac{\Delta}{A^2}\left(P+\frac{2}{9}(\Pi_{II}+2\Pi_I)\right)+\frac{\xi_4}{\kappa}\right\}+\frac{\xi_1}{4A^2}\right]+\frac{\xi_7\Delta}{A^2}.
$$

where

$$
\xi_5=T^{0\beta}_{\ ;\beta}-\left(P+\frac{2}{9}(2\Pi_{II}+\Pi_I)\right)\frac{r^2A^2B^2A'}{\Delta A}+\frac{A'\mu r^4A^4B^4}{\Delta^2A}-\left(P+\frac{2}{9}(\Pi_{II}+2\Pi_I)\right)',
$$

$$
\xi_6=\xi_2+\frac{A^2r^2B^2f'_R}{\Delta}\left(\frac{B'}{B}+\frac{1}{r}\right)-\frac{C'f'_R}{C}-\frac{L\Delta L'f'_R}{2},
$$

$$
\xi_7=T^{1\beta}_{\ ;\beta}-\frac{A^2}{\Delta}P_\theta+\frac{A_\theta}{A}\left[\frac{5r^2A^4B^2L\mu}{\Delta^2}-\frac{r^2A^2B^2}{\Delta^2}\left(P+\frac{2}{9}(\Pi_{II}+2\Pi_I)\right)\right],
$$

$$
\psi_m=\frac{(2m_{tot}-r^2B')+D\sqrt{(r^2B'-2m_{tot})^2+4r^4\dot{B}^2(rA^2B'-A^2B_\theta^2-LB_\theta\dot{B})}}{2\dot{B}^2r^3},
$$

where ψ_m is calculated by taking an assumption that reflection effects are far lesser than that produced by other scale factors in the evolution of axisymmetric system. In order to examine the contributions of f'_R and $f_{R\theta}$ across the meridional non-static axial relativistic object, we multiply both sides of the above equations with $\frac{df}{dP}$. After some manipulations, this yields couple of quadratic equations in f'_R and $f_{R\theta}$ whose solutions become

$$f'_R = \frac{1}{18f_R C r^3 A^2 \psi_m^2 \Delta}\left[-\psi_m r^2 A^2(144(f_R\psi_m C + f_R\psi_m r C' + C r\psi'_m)\right.$$
$$\left. +2\mathcal{C}_1\Delta r C\psi_m + 9\mathcal{C}_2 r C\psi_m\Delta) + D\sqrt{\Delta_1}\right], \tag{31}$$
$$f_{R\theta} = \frac{1}{72A^4B^3CL\Delta(A^2b^2r^2-L^2)}\left[-36A^4 f_R B^2 r^2 L^2\kappa(r^2B^3\dot{C} - 2CLb_\theta\right.$$
$$-2LBC_\theta) + 36A^6B^4Lr^4 f_R\kappa(BC_\theta - CB_\theta) - 36A^2r^2B^2LC\Delta^2(r^2B^2A^2$$
$$\left. -L^2) + D\sqrt{\Delta_2}\right], \tag{32}$$

where Δ_1 and Δ_2 are discriminants of f'_R and $f_{R\theta}$ quadratic equations, $\mathcal{C} = \left[\frac{9\mu r^4A^4\psi_m^4 - 9r^2A^2\psi_m^2\Delta\{9P+2(\Pi_{II}+2\Pi_I)\}}{9\Delta^2}\right]\frac{df_R}{dP}$, $\mathcal{C}_1 = 2(\Pi_{II}+2\Pi_I)'\frac{df_R}{dP}$, while $D = \pm 1$. We shall take $A_0 = 1-\varphi,\ B_0 = 1+\varphi$ with $\varphi = \frac{m_0}{r}$ for pN epochs, therefore

$$\frac{A'_0}{A_0} = (1+\varphi)'(1-\varphi), \quad \frac{A_{0\theta}}{A_0} = (1+\varphi)_\theta(1-\varphi).$$

Over the surface of axial reflection relativistic star object, Eqs.(31) and (32) yield

$$\frac{f'_R}{f_R} = \frac{W_1(D-1)}{18C\psi_m\Delta}, \quad \frac{f_{R\theta}}{f_R} = \frac{W_2(D-1)}{2}, \tag{33}$$

where $W_1 = 144(\psi_m C + \psi_m r C' + rC\psi'_m)$ and $W_2 = \frac{A^2r^2L\kappa}{(A^2B^2r^2-L^2)}(r^2\dot{C}\psi_m + 2LC\psi_{m\theta} - 2LC_\theta\psi_m) - A^4B^2r^4\kappa(BC_\theta - CB_\theta) + r^2C\Delta^2(r^2A^2B^2 - L^2)$. It can analyzed from Eq.(33) that on setting $D = -1$, one can get specific forms of γ and γ_1 from Eqs.(28) and (29) which will make $\frac{A'}{A}$ and $\frac{A_\theta}{A}$ approach to ∞ with $(r,\theta) \to (r^-,\theta^-)$, while finite value of $\frac{A'}{A}$ and $\frac{A_\theta}{A}$ can be achieved for $(r,\theta) \to (r^+,\theta^+)$. For physically viable stellar model, we take $D = 1$ which yields $f_{R\theta} = 0 = f'_R$ for $(r,\theta) \to (r^-,\theta^-)$. This reinforces the continuity of $f_{R\theta},\ f'_R$ as well as A' over the surface of axial stellar structure with reflection degrees of freedom.

4.1 Newtonian Approximation

In order to evaluate instability conditions at N regime, we take $A_0 = B_0 = 1$ and assume anisotropic pressure to be less than zero which is the criterion for collapsing celestial body. We also take configurations of initial perturbed structural coefficients to be $C_0 = L_0 = r$. Consequently, the collapse equation (25) turns out to be

$$\Gamma_1\phi'_{\mathcal{N}} + \frac{9}{4r}\phi_{\mathcal{N}}\Gamma_1 - \Gamma_1\frac{\Pi_{KN0,\theta}}{2r\sqrt{2}}\left(\frac{2c}{r} + 3b + \frac{l}{r}\right)_\theta = \frac{3}{8r}\left(\frac{2c}{r} + 3b + \frac{l}{r}\right) - 2b\left[P'_0\right.$$
$$\left.+\frac{2}{9}(2\Pi'_{I0} + \Pi'_{II0})\right] + \frac{1}{2r\sqrt{2}}\left(2b - \frac{l}{r}\right) + \frac{b}{2}X_{30\mathcal{N}} + \frac{l}{2r}X_{10\mathcal{N}} + \Upsilon + \Phi + \zeta,$$

where subscript $\mathcal{N}$ indicates the evaluation of term at N regime. We assume that all terms on both sides of the above equation are positive. The instability constraint for meridional axisymmetric fluid configurations is given by

$$\Gamma_1 < \frac{\frac{3}{8r}\left(\frac{2c}{r} + 3b + \frac{l}{r}\right) - 2b\left[P'_0 + \frac{2}{9}(2\Pi'_{I0} + \Pi'_{II0})\right] + \phi_1 + \zeta_N}{\phi'_{\mathcal{N}} + \frac{9}{4r}\phi_{\mathcal{N}} - \frac{\Pi_{KN0,\theta}}{2r\sqrt{2}}\left(\frac{2c}{r} + 3b + \frac{l}{r}\right)_\theta}, \qquad (34)$$

where $\phi_1 = \frac{b}{2}X_{30\mathcal{N}} + \frac{l}{2r}X_{10\mathcal{N}} + \frac{1}{2r\sqrt{2}}\left(2b - \frac{l}{r}\right) + \Upsilon_{\mathcal{N}} + \Phi_{\mathcal{N}} + \zeta_{\mathcal{N}}$. The system would be in complete hydrostatic equilibrium, if (during evolution) it can take a value equal to the right hand side of the above expression. However, on satisfying the above inequality, the system will move in the unstable phase. This constraint is being mentioned through Γ_1 parameter thereby emphasizing the importance of matter stiffness factor in our investigation.

4.2 Post-Newtonian Approximation

Here, we take axial structural coefficients for pN eras and consider our outcomes upto $O(\varphi)$. Using these relations in Eq.(25), one can have modified collapse equation at pN limit. This leads to instability inequality through stiffness parameter

$$\Gamma_1 < \frac{r^2(1-4\phi)\{\varphi' + \frac{1}{r}(1-\varphi)(1-\varphi)_\theta\}\chi_{pN} + (1-2\varphi)\frac{\chi_{pN}}{4}(\varphi' + \frac{3}{2r}) + \zeta_1}{(1-2\varphi)\phi'_{pN} + (1-2\varphi)\phi_{pN}[\frac{7}{4r} + \frac{1}{2}(\frac{1}{r} - \varphi')] + \zeta_2}, (35)$$

where

$$\zeta_1 = \Pi_{KN0\theta} - 2b(1-3\varphi)\left[P'_0 + \frac{2}{9}(2\Pi'_{I0} + \Pi'_{II0})\right] + \frac{(1+2\varphi)}{r\sqrt{2}}(3b - a + 2a\varphi$$

$$-3b\varphi-\frac{3l}{r}\bigg)+\frac{l}{2r}X_{10pN}+\frac{b(1-\varphi)}{2}X_{30pN}+\Upsilon_{pN}+\Phi_{pN}+\zeta_{pN},$$

$$\zeta_2=-\frac{r^3(1-\varphi)}{2\sqrt{2}}\frac{\Pi_{KN0}\chi_{pN}}{\mu_0+\Pi_{KN0}}[6(1+\varphi)_\theta(1-\varphi)+(1+\varphi)_\theta(1+\varphi)+2(1+\varphi)$$

$$\times\{(1-\varphi)_\theta(1+\varphi)+(1+\varphi)_\theta(1-\varphi)\}]+\frac{(1+2\varphi)}{2\sqrt{2}}\left(\frac{\Pi_{KN0}\chi_{pN}}{\mu_0+\Pi_{KN0}}\right)_\theta,$$

the subscript pN represents effects of quantities at pN era. The quantity Υ_{pN} describes the reflection effects of non-static axial celestial body about its symmetry axis at pN approximations. It is worth mentioning here that these constraints coincide with [27] in the limit $L\to 0$ for $n=2$.

5 Instability of Realistic Star Object

Perturbations of stars and black holes have been one of the main topics of relativistic astrophysics for the last few decades. The description of such stellar objects has recently attracted various researchers [38]. The stability analysis of general relativistic star process is an important but challenging endeavor. In such study, the spherical symmetric matter configuration is an exemplary one. Numerous realistic objects like globular clusters, galactic bulges and dark matter haloes can be considered as being roughly spherical geometry. For better understanding of cosmic censorship hypothesis and hoop conjecture, it is necessary to throw light on non-spherical collapse. The physical interest in studying non-spherical symmetries is associated with the fact that post-shocked clouds are left at the verge of gravitational collapse forming cylinders or plates at scales of galaxy formation and at scales of stellar formation in galaxy. For instance, cylindrical distributions are closely related with the problem of fragmentation of prestellar clouds [39].

We take into account a specific configurations of non-static axial spacetime. The main purpose is to study instability regimes of axially symmetric realistic objects that are involved in the emission of gravitational radiation due to meridional degrees of freedom. For this purpose, we assume coupling of system with anisotropic fluid distribution whose energy-momentum tensor is mentioned in Eq.(5). Having arrived at this point, the relevant question is: does an ideal (or non anisotropic) matter configuration produce gravitational radiations?

To answer such a burning issue, let us recall that in the seminal paper of

Bondi about the emission of gravitational radiation (section 6 of [40]), it is mentioned that for relativistic dust cloud as well as dissipation-less case of an ideal matter distribution, the relativistic system cannot be anticipated to radiate (gravitationally). This is due to the reversible feature of equation of state as emission of radiation is an irreversible phenomenon. This happens once when absorption is considered (and/or Sommerfeld type constraints), which prevents inflow of waves. This implies that an entropy generator parameter/factor must be present in the discussion of relativistic source. However, such type of factor is not present in an ideal fluid and in a collisionless dust cloud. In particular, the irreversibility of gravitational wave emissions must be taken in equation of state with the help of an entropy increasing (dissipative) parameter. In this scenario, Herrera *et al.* [41] described a close relationship between vorticity and gravitational radiations.

We consider the evolution of non-static axisymmetric self-gravitating system in $f(R)$ gravity and assume that it is in hydrostatic equilibrium at large past time. Now we want to analyze that when the phase of equilibrium is disturbed, what happens? Will this perturbation be relaxed (stable state) or will it grow (unstable state). In this respect, one needs to take into account couple of following instabilities

1. dynamical stability: what happens, if stellar hydrostatic phase is perturbed?

2. secular (thermal) stability: what happens, when the state of thermal equilibrium is perturbed?

Since our system is coupled with anisotropic matter configurations without heat flux, therefore we shall not discuss the second case and confine ourselves over dynamical instability of relativistic origin. It is seen that under hydrostatic phase, the stability criterion is achieved by making linearized field as well as conservation equations against radial perturbation (14)-(17). It is remarked that during evolution, the realistic object moves via several evolutionary patterns determined by instability/stability degrees of freedom. This suggests that the relativistic systems can be stable at one instance but not at the other. Thus one needs to cope with the dynamical evolution of self-gravitating systems by calculating instability regions at N as well as pN regimes. Such epochs have vital role in the discussion of gravitational collapse of compact objects.

The phenomenon of celestial collapse occurs when the state of hydrostatic equilibrium of a stellar object is disturbed. In celestial body, nuclear fission reactions occur that start from hydrogen atoms and produce further complex elements until nuclear reactions chain stops with iron. These reactions increase the pressure exerted by gas particles which counterbalance the gravitational attraction and prevents the star from collapsing. However, with the passage of time, nuclear reactions decrease as fuel burns out. Consequently, the necessary pressure becomes insufficient for a collapsing body to be stable. At this point, the gravitational force begins to pull matter towards the center of a body and thus collapse initiates. A celestial body that has exhausted all its nuclear fuel, can give birth to three possible compact objects (white dwarfs, neutron stars and black holes) on the basis of the initial mass of the collapsing body.

It is well-known that, in the scenario of Newtonian regime, the instability of spherical self-gravitating systems depends purely on the mean value of stiffness parameter, Γ_1 [42] which is the ratio of fractional Lagrangian variations between pressure and energy density experienced by matter configurations following the motion. However, in GR, the stability relies not only on the average value of Γ_1 but also on the star radius. However, in the study of non-static axial reflection system in modified gravity, the situation is quite different. (It is worthy to stress that we have assumed Γ_1 as a constant entity throughout the analysis). The most important consequence of our study is that, apart from affirming GR results, Γ_1 controls emission of gravitational radiations along with $f(R)$ extra degrees of freedom. The emission of gravitational radiations causes the loss of both energy as well as angular momentum which increases the instability of the meridional axisymmetric object.

More specifically, following the results of Chandrasekhar [15], we deduce that if the anisotropic matter distribution attains stiffness equal to the right hand side of expressions (34) and (35), the system enters into the window of hydrostatic equilibrium at N and pN regimes. Further, if the stiffness of fluid increases in such a way that the fractional value given at the right hand side of (34) and (35) becomes a smaller one, then system enters into the stable configurations at both N and pN approximations, respectively. Dosopoulou *et al.* [43] explored the contribution of magnetic fields in the emergence and existence of vorticity. This strongly suggests that invoking of magnetic fields in the study of stability of gravitationally radiating sources deserves attention for future work.

6 Conclusions

It is well-known that the most general non-static axial geometry incorporates reflection (meridional) and rotation effects coming out from non-diagonal $dtd\theta$ and $dtd\phi$ metric coefficients. In order to dealt analytically with instability constraints of axially symmetric spacetime, several attempts have been made by taking restricted class of axial geometry. In this paper, we have studied stability analysis of meridional axial stellar structure with $f(R)$ background. We are observing investigation in a metric $f(R)$ gravity which give rise to non-linear fourth order field equations. We have formulated the collapse equation by using perturbation scheme in the generalized Euler equation. We assume complete hydrostatic equilibrium of axial stellar structure at large past time, i.e., $T(-\infty) = 0$.

We have developed instability constraints at N and pN epochs through stiffness parameter, Γ_1 using collapse equation. It is found that axial stellar structure would be unstable until it obeys relation (34) at N regime while relation (35) at pN era. Breaching of these inequalities will eventually move the system towards stable window. These constraints depend upon adiabatic index, static combinations of anisotropy, energy density and dark source corrections due to $R+\epsilon R^n$ model. It is seen that dark source corrections tend to stabilize structure formation phenomenon due to its non-attractive behavior while the presence of non-diagonal terms in instability ranges indicate occurrence of gravitational radiations which correspond to flow of super-energy [14].

We have found non-vanishing component of vorticity tensor which corresponds to non-static meridional axial structure coefficient. The inclusion of non-diagonal scale factor in the stability analysis leads to interesting phenomenon of gravitational radiations for ϵR^n corrections. These extra-order $f(R)$ corrections affect the passive gravitational mass which in turn affect the rate of stellar collapse. We have developed instability constraints (34) and (35) with weak field and pN approximations, respectively. These constraints can be applied to axisymmetric self-gravitating system with reflection degrees around symmetry axis at some particular cosmic epochs depending upon the chosen values of n. We can categorize different eras of cosmic dynamics associated with ϵR^n as follows.

- For $n = 2$, the instability constraints for specific model of the type $R + \epsilon R^2$ can be obtained. The existence of R^2 correction in the field

equations can be helpful to explain inflationary mechanism of cosmos. The term αR^2 represents accelerated expansion of the universe. This model is compatible with temperature anisotropies noticed in cosmic microwave background radiations [31] and hence viable for inflationary scalar field models.

- The choice $n = 3$ favors to host significant massive compact objects coming out from cubic $f(R)$ higher curvature terms [24]. This provides realistic signature of the presence of more massive and huge self-gravitating stellar systems which have direct correspondence with the observational cosmology.

- This gravitational dynamics at late-time universe era can be obtained by substituting $n = -1$ in instability constraints at both N and pN regimes. It is noticed that gravitational contribution due to negative curvature power serves as dark energy thereby supporting current accelerating cosmic epochs [44].

- For $\epsilon = 0$, instability constraints for Einstein gravity can be obtained at both N and pN eras which describes relatively less stable axial stellar structure.

Finally, we remark that supermassive stellar systems survive more abundantly in extended gravity than in GR as such theories (for instance $f(R)$ gravity) are more likely to host huge stars with smaller radii. This leads to the existence of more dense relativistic systems which have direct relevance with observational gravitational physics. It is interesting to mention here that all our results reduce to restricted class of instability analysis [27] by neglecting non-diagonal terms and assuming $n = 2$ in $f(R)$ model.

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Appendix A

The extra $f(R)$ curvature terms for Eqs.(10)-(12) are

$$
\begin{aligned}
D_0 &= \dot{V}_1 + X_1' + X_3' + \left(\frac{3B^2r^2AA'}{\Delta} + \frac{2A^2B^2r}{\Delta} + \frac{2r^2A^2BB'}{\Delta} + \frac{C'}{C} + \frac{B'}{B}\right)X_1 \\
&+ \left(\frac{3r^2B^2AA_\theta}{\Delta} + \frac{B_\theta}{B} + \frac{C_\theta}{C} + \frac{A^2r^2BB_\theta}{\Delta}\right)X_3 + \left(\frac{r^2B^3\dot{B}}{\Delta} + \frac{B'}{B}\right)V_2 \\
&+ \frac{r^2B^2}{\Delta}(r^2B\dot{B}V_3 + C\dot{C}V_4) + \dot{W}_1 + (Y_1 + Y_3)' + \frac{LW_1}{\Delta}(AA_\theta + L\dot{L}) + \frac{L\dot{L}V_1}{\Delta} \\
&+ \left(\frac{3r^2B^2AA'}{\Delta} + \frac{B'}{B} + \frac{2rA^2B^2}{\Delta} + \frac{2r^2A^2BB'}{\Delta} + \frac{C'}{C} + \frac{4LL'}{\Delta}\right)Y_1\left(\frac{B_\theta}{B}\right. \\
&\times \frac{3r^2B^2AA_\theta}{\Delta} + \frac{r^2A^2BB_\theta}{\Delta} + \frac{C_\theta}{C} + \frac{3Lr^2B\dot{B}}{\Delta} - \frac{r^2B^2L'}{2\Delta} + \frac{rLB^2}{\Delta} + \frac{r^2LBB'}{\Delta} \\
&\left.-\frac{r^2LB\dot{B}}{\Delta} + \frac{LL_\theta}{\Delta}\right)Y_3 + \left(\frac{3r^2LB\dot{B}}{\Delta} - \frac{r^2B^2L'}{2\Delta} + \frac{r^2LBB'}{\Delta} + \frac{rB^2L}{\Delta} + \frac{LL_\theta}{\Delta}\right. \\
&\left.-\frac{r^2LB\dot{B}}{\Delta}\right)X_3 + \left(\frac{r^2B^3\dot{B}}{\Delta} + \frac{B'}{B} - \frac{LBB_\theta}{\Delta}\right)W_2 - \frac{LBB_\theta}{\Delta}V_2 + \left(\frac{rLB^2}{\Delta}\right. \\
&\left.+\frac{r^2BLB'}{\Delta} - \frac{r^2B^2L'}{2\Delta}\right)(X_2 + Y_2) + \left(\frac{rBLB_\theta}{\Delta} - \frac{r^2B^2L_\theta}{\Delta} + \frac{r^4B^3\dot{B}}{\Delta}\right)W_3 \\
&+ \left(\frac{rBLB_\theta}{\Delta} - \frac{r^2B^2L_\theta}{\Delta}\right)V_3 + \left(\frac{r^2}{\Delta}B^2C\dot{C} - \frac{LCC_\theta}{\Delta}\right)W_4 - \frac{LCC_\theta}{\Delta}V_4,
\end{aligned}
\tag{A1}
$$

$$
\begin{aligned}
D_1 &= \dot{X}_1 - \left(r + \frac{r^2B'}{B}\right)W_3 - \frac{CC'}{B^2}W_4 + X_{2\theta} + V_2' + \frac{AA'}{B^2}V_1 + \left(\frac{2B'}{B}\right. \\
&\left.+\frac{r^2B^2AA'}{\Delta} + \frac{2rA^2B^2}{\Delta} + \frac{2r^2A^2BB'}{\Delta} + \frac{C'}{C}\right)V_2 + \left(\frac{3B_\theta}{B} + \frac{r^2B^2AA_\theta}{\Delta}\right. \\
&\left.+\frac{r^2A^2BB_\theta}{\Delta} + \frac{C_\theta}{C}\right)X_2 + \frac{A^2r^2B\dot{B}}{\Delta}X_3 - \left(r + \frac{r^2B'}{B}\right)V_3 - \frac{CC'V_4}{B^2} + \dot{Y}_1 \\
&+ W_2' + Y_{2\theta} + \left(\frac{3\dot{B}}{B} + \frac{r^2B^2A\dot{A}}{\Delta} + \frac{\dot{C}}{C} + \frac{LAA_\theta}{\Delta} + \frac{L\dot{L}}{\Delta}\right)Y_1 + \left(\frac{LAA_\theta}{\Delta}\right.
\end{aligned}
$$

$$+\frac{L\dot{L}}{\Delta}\Bigg)X_1+\left(\frac{2B'}{B}+\frac{r^2B^2AA'}{\Delta}+\frac{2rA^2B^2}{\Delta}+\frac{2r^2A^2BB'}{\Delta}+\frac{C'}{C}+\frac{3LL'}{\Delta}\right)W_2$$
$$+\frac{3LL'}{\Delta}V_2+\left(\frac{3B_\theta}{B}+\frac{r^2B^2AA_\theta}{\Delta}+\frac{r^2B^2BB_\theta}{\Delta}+\frac{C_\theta}{C}+\frac{LL_\theta}{\Delta}\right)Y_2+\frac{LL_\theta}{\Delta}X_2$$
$$+\left(\frac{r^2A^2B\dot{B}}{\Delta}-\frac{L'}{B}-\frac{LAA_\theta}{\Delta}\right)Y_3-\left(\frac{L'}{B}+\frac{LAA_\theta}{\Delta}\right)X_3, \qquad \text{(A2)}$$
$$D_2=\dot{X}_3+X'_2+V_{3\theta}+\frac{A^3A_\theta}{\Delta}V_1-\frac{A^2BB_\theta}{\Delta}V_2-\frac{A^2CC_\theta}{\Delta}V_4+\Bigg(\frac{r^2B^2AA_\theta}{\Delta}$$
$$+\frac{2r^2A^2BB_\theta}{\Delta}+\frac{B_\theta}{B}+\frac{C_\theta}{C}\Bigg)V_3+\Bigg(\frac{3A^2r^2B\dot{B}}{\Delta}-\frac{2ALA_\theta}{\Delta}+\frac{r^2B^2A\dot{A}}{\Delta}$$
$$+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\Bigg)X_3+Y'_2+\dot{Y}_3+\Bigg(\frac{6rA^2B^2}{\Delta}+\frac{6r^2A^2BB'}{\Delta}+\frac{2r^2B^2AA'}{\Delta}+\frac{B'}{B}$$
$$+\frac{C'}{C}\Bigg)X_2+W_{3\theta}+\left(\frac{A^3A_\theta}{\Delta}+\frac{A^2\dot{L}}{\Delta}-\frac{AL\dot{A}}{\Delta}\right)W_1+\left(\frac{A^2\dot{L}}{\Delta}-\frac{AL\dot{A}}{\Delta}\right)V_1$$
$$-\left(\frac{A^2BB_\theta}{\Delta}-\frac{BL\dot{B}}{\Delta}\right)W_2-\frac{BL\dot{B}}{\Delta}V_2-\left(\frac{A^2}{\Delta}CC_\theta+\frac{CL\dot{C}}{\Delta}\right)W_4$$
$$-\frac{CL\dot{C}}{\Delta}V_4+\Bigg(\frac{r^2B^2AA_\theta}{\Delta}+\frac{2r^2A^2BB_\theta}{\Delta}+\frac{r^2BL\dot{B}}{\Delta}+\frac{2LL_\theta}{\Delta}+\frac{B_\theta}{B}+\frac{C_\theta}{C}$$
$$-\frac{2r^2LB\dot{B}}{\Delta}\Bigg)W_3+\frac{LV_3}{\Delta}(r^2BL\dot{B}+2L_\theta-2r^2B\dot{B})+\left(\frac{A^2L'}{\Delta}-\frac{2ALA'}{\Delta}\right)$$
$$\times(X_1+Y_1)+\left(\frac{3A^2r^2B\dot{B}}{\Delta}-\frac{2ALA_\theta}{\Delta}+\frac{r^2B^2A\dot{A}}{\Delta}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}+\frac{L\dot{L}}{\Delta}\right)Y_3$$
$$+\left(\frac{L\dot{L}}{\Delta}-\frac{2LAA_\theta}{\Delta}\right)X_3+\Bigg(\frac{6rA^2B^2}{\Delta}+\frac{6r^2A^2BB'}{\Delta}+\frac{2r^2B^2AA'}{\Delta}+\frac{B'}{B}$$
$$+\frac{C'}{C}+\frac{7LL'}{2\Delta}\Bigg)Y_2+\frac{7LL'}{2\Delta}X_2. \qquad \text{(A3)}$$

The perturbed parts of Eq.(10) are

$$g=\Bigg[x'_1+x_1\left(\frac{3B_0^2r^2A_0A'_0}{\Delta_0}+\frac{B'_0}{B_0}+\frac{2A_0^2B_0^2r}{\Delta_0}+\frac{2A_0^2r^2B_0B'_0}{\Delta_0}+\frac{C'_0}{C_0}\right)+X_{10}$$

$$\times\left\{\frac{3B_0^2r^2A_0A_0'}{\Delta_0}\left(\frac{2b}{B_0}+\frac{a}{A_0}+\frac{a'}{A_0'}-\frac{d}{\Delta_0}\right)+\left(\frac{b}{B_0}\right)'+\frac{2A_0^2B_0^2r}{\Delta_0}\left(\frac{2a}{A_0}+\frac{2b}{B_0}\right.\right.$$
$$\left.-\frac{d}{\Delta_0}\right)+\frac{2A_0^2r^2B_0B_0'}{\Delta_0}\left(\frac{2a}{A_0}+\frac{b}{B_0}+\frac{b'}{B_0'}\right)+\left.\left(\frac{c}{C_0}\right)'\right\}+x_3\left(\frac{3r^2B_0^2A_0A_{0\theta}}{\Delta_0}\right.$$
$$\left.+\frac{C_{0\theta}}{C_0}+\frac{B_{0\theta}}{B_0}+\frac{A_0^2r^2B_0^2B_{0\theta}}{\Delta_0}\right)+X_{30}\left\{\frac{3r^2B_0^2A_0A_{0\theta}}{\Delta_0}\left(\frac{2b}{B_0}+\frac{a}{A_0}+\frac{a_\theta}{A_{0\theta}}\right.\right.$$
$$\left.\left.-\frac{d}{\Delta_0}\right)+\left(\frac{c}{C_0}\right)_\theta+\left(\frac{c}{C_0}\right)_\theta+\frac{A_0^2r^2B_0^2B_{0\theta}}{\Delta_0}\left(\frac{2a}{A_0}+\frac{2b}{B_0}+\frac{b_\theta}{B_{0\theta}}\right)\right\}$$
$$+W_{10}\frac{L_0A_0A_{0\theta}}{\Delta_0}\left(\frac{l}{L_0}+\frac{a}{A_0}+\frac{a_\theta}{A_{0\theta}}-\frac{d}{\Delta_0}\right)+y_1\left(\frac{3r^2B_0^2A_0^2B_{0\theta}}{\Delta_0}+\frac{C_0'}{C_0}+\frac{B_0'}{B_0}\right.$$
$$\left.+\frac{2A_0^2B_0^2r}{\Delta_0}+\frac{2A_0^2B_0r^2B_0'}{\Delta_0}+\frac{4L_0L_0'}{\Delta_0}\right)+Y_{10}\left\{\frac{3r^2B_0^2A_0^2B_{0\theta}}{\Delta_0}\left(\frac{2b}{B_0}+\frac{a}{A_0}+\frac{a'}{A_0'}\right.\right.$$
$$\left.-\frac{d}{\Delta_0}\right)+\left(\frac{b}{B_0}\right)'+\frac{2A_0^2B_0^2r}{\Delta_0}\left(\frac{2a}{A_0}+\frac{2b}{B_0}-\frac{d}{\Delta_0}\right)+\frac{2A_0^2B_0r^2B_0'}{\Delta_0}\left(\frac{2a}{A_0}+\frac{b}{B_0}\right.$$
$$\left.\left.+\frac{b'}{B_0'}-\frac{d}{\Delta_0}\right)+\left(\frac{c}{C_0}\right)'+\frac{4L_0L_0'}{\Delta_0}\left(\frac{l}{L_0}+\frac{l'}{L_0'}-\frac{d}{\Delta_0}\right)\right\}+\frac{4x_1L_0L_0'}{\Delta_0}+X_{10}$$
$$\times\frac{4L_0L_0'}{\Delta_0}\left(\frac{l}{L_0}+\frac{l'}{L_0}-\frac{d}{\Delta_0}\right)+y_3\left(\frac{3r^2B_0^2A_0A_{0\theta}}{\Delta_0}+\frac{C_{0\theta}}{C_0}+\frac{A_0^2r^2B_0B_{0\theta}}{\Delta_0}\right.$$
$$\left.+\frac{B_{0\theta}}{B_0}-\frac{r^2L_0'B_0^2}{2\Delta_0}+\frac{rL_0B_0^2}{\Delta_0}+\frac{r^2L_0B_0B_0'}{\Delta_0}+\frac{L_0L_{0\theta}}{\Delta_0}\right)+Y_{30}\left\{\frac{3r^2B_0^2A_0A_{0\theta}}{\Delta_0}\right.$$
$$\times\left(\frac{2b}{B_0}+\frac{a}{A_0}+\frac{a_\theta}{A_{0\theta}}-\frac{d}{\Delta_0}\right)+\left(\frac{b}{B_0}\right)_\theta+\frac{A_0^2r^2B_0B_{0\theta}}{\Delta_0}\left(\frac{b}{B_0}+\frac{2a}{A_0}+\frac{b_\theta}{B_{0\theta}}\right.$$
$$\left.-\frac{d}{\Delta_0}\right)+\left(\frac{c}{C_0}\right)_\theta-\frac{r^2L_0'B_0^2}{2\Delta_0}\left(\frac{l'}{L_0'}+\frac{2b}{B_0}-\frac{d}{\Delta_0}\right)+\frac{rL_0B_0^2}{\Delta_0}\left(\frac{l}{L_0}+\frac{2b}{B_0}-\frac{d}{\Delta_0}\right)$$
$$\left.+\frac{r^2L_0B_0B_0'}{\Delta_0}\left(\frac{l}{L_0}+\frac{b}{B_0}+\frac{b'}{B_0'}-\frac{d}{\Delta_0}\right)+\frac{L_0L_{0\theta}}{\Delta_0}\left(\frac{l}{L_0}+\frac{l_\theta}{L_{0\theta}}-\frac{d}{\Delta_0}\right)\right\}+x_3$$
$$\times\left(\frac{rB_0^2L_0}{\Delta_0}-\frac{r^2B_0^2L_0'}{2\Delta_0}+\frac{r^2B_0B_0'L_0}{\Delta_0}+\frac{L_0L_{0\theta}}{\Delta_0}\right)+X_{30}\left\{\frac{rB_0^2L_0}{\Delta_0}\left(\frac{2b}{B_0}-\frac{d}{\Delta_0}\right)\right.$$
$$-\frac{r^2B_0^2L_0'}{2\Delta_0}\left(\frac{2b}{B_0}+\frac{l'}{L_0'}-\frac{d}{\Delta_0}\right)+\frac{r^2B_0B_0'L_0}{\Delta_0}\left(\frac{b}{B_0}+\frac{l}{L_0}+\frac{b'}{B_0'}-\frac{d}{\Delta_0}\right)$$
$$\left.+\frac{L_0L_{0\theta}}{\Delta_0}\left(\frac{l}{L_0}+\frac{l_\theta}{L_{0\theta}}-\frac{d}{\Delta_0}\right)\right\}+w_2\left(\frac{B_0'}{B_0}-\frac{L_0B_0B_{0\theta}}{\Delta_0}\right)+\left\{\left(\frac{b}{B_0}\right)'-\frac{L_0}{\Delta_0}\right.$$

$$\times B_0B_{0\theta}\left(\frac{l}{L_0}+\frac{b}{B_0}+\frac{b_\theta}{B_{0\theta}}-\frac{d}{\Delta_0}\right)\right\}W_{20}-V_{20}\frac{L_0B_0B_{0\theta}}{\Delta_0}\left(\frac{l}{L_0}+\frac{b}{B_0}+\frac{b_\theta}{B_{0\theta}}\right.$$
$$\left.-\frac{d}{\Delta_0}\right)-v_2\frac{L_0B_0B_{0\theta}}{\Delta_0}+(x_2+y_2)\left(\frac{r}{\Delta_0}L_0B_0^2+\frac{r^2B_0B_0'}{\Delta_0}-\frac{r^2B_0^2L_0'}{2A_0}\right)$$
$$+(x_2+y_2)\left\{\frac{rL_0B_0^2}{\Delta_0}\left(\frac{l}{L_0}+\frac{2b}{B_0}-\frac{d}{\Delta_0}\right)+\frac{r^2B_0B_0'}{\Delta_0}\left(\frac{l}{L_0}+\frac{b}{B_0}+\frac{b'}{B_0'}\right.\right.$$
$$\left.\left.-\frac{d}{\Delta_0}\right)-\frac{r^2B_0^2L_0'}{2A_0}\left(\frac{l'}{L_0'}+\frac{2b}{B_0}-\frac{d}{\Delta_0}\right)\right\}+(w_3+v_3)\left(\frac{r}{\Delta_0}B_0L_0B_{0\theta}\right.$$
$$\left.-\frac{r^2B_0^2L_{0\theta}}{\Delta_0}\right)+(W_{30}+V_{30})\left\{\frac{rB_0L_0B_{0\theta}}{\Delta_0}\left(\frac{l}{L_0}+\frac{b}{B_0}+\frac{b_\theta}{B_{0\theta}}-\frac{d}{\Delta_0}\right)-\left(\frac{2b}{B_0}\right.\right.$$
$$\left.\left.+\frac{a_\theta}{A_{0\theta}}-\frac{d}{\Delta_0}\right)\frac{r^2B_0^2L_{0\theta}}{\Delta_0}\right\}-(v_4+w_4)\frac{L_0C_0C_{0\theta}}{\Delta_0}-(V_{40}+W_{40})\frac{L_0C_0C_{0\theta}}{\Delta_0}$$
$$\left.\left(\frac{l}{L_0}+\frac{c}{C_0}+\frac{c_\theta}{C_{0\theta}}-\frac{d}{\Delta_0}\right)+V_{20}\left(\frac{b}{B_0}\right)'+v_2\frac{B_0'}{B_0}+\frac{w_1L_0A_0A_{0\theta}}{\Delta_0}\right]T, \qquad \text{(A4)}$$
$$h=v_1+V_{10}\left(\frac{B_0^2r^2A_0a}{\Delta_0}+\frac{c}{C_0}\right)+\frac{lL_0}{\Delta_0}(V_{10}+W_{10})+\frac{bB_0^3r^2}{\Delta_0}(V_{20}+W_{20})$$
$$+\frac{bB_0^3r^4}{\Delta_0}(V_{30}+W_{30})+\frac{cC_0B_0^2r^2}{\Delta_0}(V_{40}+W_{40})+w_1+\frac{L_0^2lW_{10}}{\Delta_0}. \qquad \text{(A5)}$$

The perturbed parts of Eq.(23) are

$$\zeta=\frac{r^3A_0B_0^3}{\Delta_0^{\frac{3}{2}}}\left(\frac{a}{A_0}+\frac{3b}{B_0}-\frac{3d}{\Delta_0}\right)\left\{\frac{A_{0\theta}}{A_0}+\frac{6B_{0\theta}}{B_0}+\frac{C_{0\theta}}{C_0}+\frac{4r^2A_0^2B_0^2}{\Delta_0}\left(\frac{A_{0\theta}}{A_0}+\frac{B_{0\theta}}{B_0}\right)\right\}.$$
$$+\frac{r^3A_0B_0^3}{\Delta_0^{3/2}}\Pi_{KL0}\left[\frac{6B_{0\theta}}{B_0}\left(\frac{b_\theta}{B_{0\theta}}+\frac{b}{B_0}\right)+\left(\frac{a}{A_0}\right)_\theta+\left(\frac{c}{C_0}\right)_\theta+\frac{4r^2A_0^2B_0^2}{\Delta_0^2}\left(\frac{2a}{A_0}\right.\right.$$
$$\left.\left.+\frac{2b}{B_0}-\frac{2d}{\Delta_0}\right)\left(\frac{a}{A_0}+\frac{b}{B_0}\right)_\theta\right]-\frac{\mu_0r^4A_0^4}{\Delta_0^2}\left(\frac{a}{A_0}\right)'-\left\{P_0+\frac{2}{9}(2\Pi_{I0}+\Pi_{II0})\right\}\frac{1}{B_0^2}$$
$$\times\left[\left(\frac{c}{C_0}\right)'+\frac{r^2A_0^2B_0^2}{Z_0^2}\left(\frac{2a}{A_0}+\frac{2b}{B_0}-\frac{2d}{\Delta_0}\right)\left(\frac{A_0'}{A_0}+\frac{1}{r}\right)+\frac{r^2A_0^2B_0^2}{\Delta_0}\left(\frac{a}{A_0}+\frac{b}{B_0}\right)'\right]$$
$$-\frac{2b}{B_0^2}\left\{P_0+\frac{2}{9}(2\Pi_{I0}+\Pi_{II0})\right\}\left[\frac{C_0'}{C_0}+\frac{r^2A_0^2B_0^2}{\Delta_0}\left(\frac{A_0'}{A_0}+\frac{1}{r}\right)\right], \qquad \text{(A6)}$$
$$\Upsilon=-\left[\frac{4r^3A_0L_0B_0^3L_{0\theta}'}{\Delta^{5/2}}\left(\frac{a}{A_0}+\frac{3b}{B_0}-\frac{3d}{\Delta_0}\right)+\frac{4L_0L_{0\theta}}{\Delta}\left(\frac{l}{L_0}+\frac{l_\theta}{L_{0\theta}}-\frac{d}{\Delta_0}\right)-\frac{A_{0\theta}L_0}{r^2A_0B_0^2}\right.$$
$$\times\left(\frac{l}{L_0}+\frac{a_\theta}{A_{0\theta}}-\frac{a}{A_0}-\frac{2b}{B_0}\right)+\frac{\mu_0L_0^2A_0^2r^2}{\delta_0^2}\left(\frac{2l}{L_0}+\frac{2a}{A_0}-\frac{2d}{\Delta_0}\right)+\frac{\mu_0L_0^2A_0^2r^2}{\Delta_0^2}\left\{\frac{L_0'}{2L_0}\right.$$

$$
\begin{aligned}
&\times\left(\frac{l'}{L_0'}-\frac{l}{L_0}\right)+\left(\frac{b}{B_0}\right)'\Bigg\}-\frac{3L_0L_0'}{2B_0^2\Delta_0}\left\{P_0+\frac{2}{9}(2\Pi_{I0}+\Pi_{II0})\right\}\left(\frac{l}{L_0}+\frac{l'}{L_0'}-\frac{d}{\Delta_0}\right)\\
&+Y_{10}\left(\frac{3b}{B_0}+\frac{ar^2A_0B_0^2}{\Delta_0}+\frac{c}{C_0}\right)+y_1\Bigg]+\left[\frac{r^2B_0L_0b}{\Delta_0}+\frac{L_0L_{0\theta}}{\Delta_0}\left(\frac{l}{L_0}+\frac{l_\theta}{L_{0\theta}}-\frac{d}{\Delta_0}\right)\right.\\
&\left.-\frac{r^2L_0bB_0}{\Delta_0}\right]\omega X_{20}-\frac{\omega r^2A_0^2bB_0Y_{30}}{\Delta_0}+\frac{A_0L_0A_{0\theta}}{\Delta_0}\left(\frac{a_\theta}{A_{0\theta}}+\frac{a}{A_0}+\frac{l}{L_0}-\frac{d}{\Delta_0}\right)Y_{10}+\frac{L_0}{\Delta_0}\\
&\times A_0A_{0\theta}(y_1+x_1)+X_{10}\frac{L_0A_0A_{0\theta}}{\Delta_0}\left(\frac{a_\theta}{A_{0\theta}}+\frac{a}{A_0}+\frac{l}{L_0}-\frac{d}{\Delta_0}\right)+w_2\left(\frac{2B_0'}{B_0}+\frac{B_0^2r^2A_0}{\Delta_0}\right.\\
&\left.\times A_0'+\frac{2rA_0^2B_0^2}{\Delta_0}+\frac{2A_0^2r^2B_0B_0'}{\Delta_0}+\frac{C_0'}{C_0}+\frac{3L_0L_0'}{\Delta_0}\right)+W_{20}\left\{2\left(\frac{b}{B_0}\right)'+\frac{B_0^2r^2A_0A_0'}{\Delta_0}\right.\\
&\times\left(\frac{a}{A_0}+\frac{a'}{A_0'}+\frac{2b}{B_0}-\frac{d}{\Delta_0}\right)+\frac{2rA_0^2B_0^2}{\Delta_0}\left(\frac{2b}{B_0}+\frac{2a}{A_0}-\frac{d}{\Delta_0}\right)+\frac{2r^2A_0^2B_0B_0'}{\Delta_0}\left(\frac{2a}{A_0}\right.\\
&\left.\left.+\frac{b'}{B_0'}+\frac{b}{B_0}-\frac{d}{\Delta_0}\right)+\left(\frac{c}{C_0}\right)'+\frac{6L_0L_0'}{\Delta_0}\left(\frac{l}{L_0}-\frac{d'}{\Delta_0}+\frac{l'}{L_0'}\right)\right\}+\frac{3w_2L_0L_0'}{\Delta_0}+y_2\\
&\times\left(\frac{3B_{0\theta}}{B_0}+\frac{C_{0\theta}}{C_0}+\frac{r^2B_0^2A_0A_{0\theta}}{\Delta_0}+\frac{r^2A_0^2B_0B_{0\theta}}{\Delta_0}+\frac{L_0L_{0\theta}}{\Delta_0}\right)+y_{2\theta}+Y_{20}\left\{3\left(\frac{b}{B_0}\right)_\theta\right.\\
&+\frac{r^2B_0^2A_0A_{0\theta}}{\Delta_0}\left(\frac{a}{A_0}+\frac{a_\theta}{A_{0\theta}}+\frac{2b}{B_0}-\frac{d}{\Delta_0}\right)+\frac{A_0^2r^2B_0B_{0\theta}}{\Delta_0}\left(\frac{2a}{A_0}+\frac{b}{B_0}+\frac{b_\theta}{B_{0\theta}}-\frac{d}{\Delta_0}\right)\\
&\left.+\left(\frac{c}{C_0}\right)_\theta+\frac{L_0L_{0\theta}}{\Delta_0}\left(\frac{l}{L_0}+\frac{l_\theta}{L_{0\theta}}-\frac{d}{\Delta_0}\right)\right\}+x_2\frac{L_0L_{0\theta}}{\Delta_0}-y_3\left(\frac{L_0'}{B_0^2}+\frac{L_0A_0A_{0\theta}}{\Delta_0}\right)\\
&+w_2'-Y_{30}\left\{\frac{L_0'}{B_0^2}+\frac{L_0A_0A_{0\theta}}{\Delta_0}\left(\frac{a}{A_0}+\frac{a_\theta}{A_{0\theta}}+\frac{l}{L_0}-\frac{d}{\Delta_0}\right)\right\}-r^2W_{30}\left(\frac{b}{B_0}\right)'-x_3\\
&\times\left(\frac{L_0'}{B_0^2}+\frac{L_0A_0A_{0\theta}}{\Delta_0}\right)-X_{30}\left\{\frac{L_0'}{B_0^2}\left(\frac{l'}{L_0'}-\frac{2b}{B_0}\right)+\frac{L_0A_0A_{0\theta}}{\Delta_0}\left(\frac{l}{L_0}+\frac{a}{A_0}+\frac{a_\theta}{A_{0\theta}}\right.\right.\\
&\left.\left.-\frac{d}{\Delta_0}\right)\right\}-w_3\left(r+\frac{r^2B_0'}{B_0}\right)-w_4\frac{C_0C_0'}{B_0^2}+W_{40}\frac{C_0C_0'}{B_0^2}\left(\frac{c}{C_0}+\frac{c'}{C_0'}-\frac{2b}{B_0}\right)-\omega\left[2y_1\right.\\
&\left.+Y_{10}\left(\frac{3b}{B_0}+\frac{ar^2A_0B_0^2}{\Delta_0}+\frac{c}{C_0}\right)\right],\qquad\qquad\text{(A7)}
\end{aligned}
$$

$$
\begin{aligned}
\Phi&=x_{2\theta}+v_2'+\frac{A_0A_0'}{B_0}\left(\frac{a'}{A_0'}+\frac{a}{A_0}-\frac{2b}{B_0}\right)V_{10}+\frac{A_0A_0'v_1}{B_0}+\left(\frac{2B_0'}{B_0}+\frac{B_0^2r^2A_0A_0'}{\Delta_0}\right.\\
&\left.+\frac{C_0'}{C_0}+\frac{2rA_0^2B_0^2}{\Delta_0}+\frac{2r^2A_0^2B_0B_0'}{\Delta_0}\right)v_2+V_{20}\left\{2\left(\frac{b}{B_0}\right)'+\frac{A_0B_0^2r^2A_0'}{\Delta_0}\left(\frac{a}{A_0}+\frac{a'}{A_0'}\right.\right.
\end{aligned}
$$

$$+\frac{2b}{B_0}-\frac{d}{\Delta_0}\right)+\frac{2rA_0^2B_0^2}{\Delta_0}\left(\frac{2b}{B_0}+\frac{2a}{A_0}-\frac{d}{\Delta_0}\right)+\frac{2rA_0^2B_0^2}{\Delta_0}\left(\frac{2a}{A_0}+\frac{b'}{B_0'}+\frac{b}{B_0}-\frac{d}{\Delta_0}\right)$$

$$+\left(\frac{c}{C_0}\right)'\right\}+x_2\left(\frac{3B_{0\theta}}{B_0}+\frac{rB_0^2A_0A_{0\theta}}{\Delta_0}+\frac{r^2A_0^2B_0B_{0\theta}}{\Delta_0}+\frac{C_{0\theta}}{C_0}\right)+X_{20}\left\{3\left(\frac{b}{B_0}\right)_\theta\right.$$

$$+\frac{r^2A_0A_{0\theta}B_0^2}{\Delta_0}\left(\frac{a}{A_0}+\frac{a_\theta}{A_{0\theta}}+\frac{2b}{B_0}-\frac{d}{\Delta_0}\right)+\left(\frac{c}{C_0}\right)_\theta+\frac{r^2A_0^2B_0B_{0\theta}}{\Delta_0}\left(\frac{2a}{A_0}+\frac{b_\theta}{B_\theta}\right.$$

$$\left.\left.+\frac{b}{B_0}-\frac{d}{\Delta_0}\right)\right\}-v_3\left(r+\frac{r^2B_0'}{B_0}\right)-r^2v_3\left(\frac{b}{B_0}\right)'-v_4\frac{C_0C_0'}{B_0^2}-\frac{C_0C_0'}{B_0^2}V_{40}\left(\frac{c'}{C_0'}\right.$$

$$\left.+\frac{c}{C_0}-\frac{2b}{B_0}\right)-\omega X_{10}\left[1+\left(\frac{3b}{B_0}+\frac{ar^2A_0B_0^2}{\Delta_0}+\frac{c}{C_0}\right)\right]. \quad \text{(A8)}$$

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